Chapter 1

Linearni diferencni rovnice

Postup:

(A) Nalezeni obecneho homogenniho reseni $y_h(n)$. Nebo tez prostor reseni homogenni rovnice, ci baze tohoto prostoru.

(B) Nalezeni partikularniho reseni $y_p(n)$ pomoci metody specialni prave strany.

(C) Nalezeni obecneho reseni $y(n) = y_h(n) + y_p(n)$. V pripade pocatecni podminky, nalezeni reseni splnujiciho pocatecni podminku.

1.1 Homogenni rovnice

Postup:

(i) Nalezeni charakteristickeho polynomu $\chi(t)$.

(ii) Nalezeni korenu charakteristickeho polynomu $\chi(t)$ splolu s nasobnosti techto korenu.

(iii) Nalezeni baze prostoru reseni homogenni rovnice pomoci vety o tvaru fundamentálního systému řešení homogenní lineární diferenční rovnice k-tého řádu s konstantními koeficienty (V3). Prvky baze maji tvar n-tych mocnin korenu charakteristickeho polynomu. V pripade vycenasobneho korenu se pak jeste prenasobuje n-kem (pripadne vyssi mocninou n-ka).

(iv) Zapsani $y(n)$ jakozto linearni kombinace prvku fundamentalniho systemu z bodu (iii). V pripade pocatecnich podminek, nalezeni reseni vyhovujiciho pocatecnim podminkam.

1.1.1 $y(n+2) + 4y(n+1) + 4y(n) = 0$

(i) $\chi(t) = t^2 + 4t + 4$,

(ii) $\{-2, -2\}$,

(iii) $\{(-2)^n, n(-2)^n\}$,

(iv) $y(n) = a(-2)^n + bn(-2)^n$, $a, b \in \mathbb{R}$.

1.1.2 $y(n+2)-3y(n+1)+2y(n)=0$

(i) $\chi(t)=t^2-3t+2$,

(ii) $\{1,2\}$,

(iii) $\{1,2^n\}$,

(iv) $y(n)=a+b2^n,\ a,b\in\mathbb{R}$.

1.1.3 $y(n+2)-6y(n+1)+13y(n)=0$

(i) $\chi(t)=t^2-6t+13$,

(ii) $\{3\pm 2i\}$,

(iii) $\left\{13^{\frac{n}{2}}\cos\left(\arctan\left(\frac{2}{3}\right)n\right),13^{\frac{n}{2}}\sin\left(\arctan\left(\frac{2}{3}\right)n\right)\right\}$,

(iv) $y(n)=a13^{\frac{n}{2}}\cos\left(\arctan\left(\frac{2}{3}\right)n\right)+b13^{\frac{n}{2}}\sin\left(\arctan\left(\frac{2}{3}\right)n\right),\ a,b\in\mathbb{R}$.

1.1.4 $y(n+2)-2y(n+1)-3y(n)=0$, $y(1)=2$, $y(2)=1$

(i) $\chi(t)=t^2-2t-3$,

(ii) $\{-1,3\}$,

(iii) $\{(-1)^n,3^n\}$,

(iv) $y(n)=a(-1)^n+b3^n,\ a=-\frac{5}{4},b=\frac{1}{4}$.

1.1.5 $y(n+2)-y(n+1)-y(n)=0$, $y(1)=y(2)=1$

(i) $\chi(t)=t^2-t-1$,

(ii) $\left\{\frac{1+\sqrt{5}}{2},\frac{1-\sqrt{5}}{2}\right\}$,

(iii) $\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n,\left(\frac{1-\sqrt{5}}{2}\right)^n\right\}$,

(iv) $y(n)=a\left(\frac{1+\sqrt{5}}{2}\right)^n+b\left(\frac{1-\sqrt{5}}{2}\right)^n,\ a=\frac{\sqrt{5}}{5},b=-\frac{\sqrt{5}}{5}$.

1.1.6 $y(n+4)+6y(n+2)+9y(n)=0$

(i) $\chi(t)=t^4+6t^2+9$,

(ii) $\{\pm i\sqrt{3},\pm i\sqrt{3}\}$,

(iii) $\left\{3^{\frac{n}{2}}\cos\left(n\frac{\pi}{2}\right),n3^{\frac{n}{2}}\cos\left(n\frac{\pi}{2}\right),3^{\frac{n}{2}}\sin\left(n\frac{\pi}{2}\right),n3^{\frac{n}{2}}\sin\left(n\frac{\pi}{2}\right)\right\}$,

(iv) $y(n)=3^{\frac{n}{2}}\left((a+bn)\cos\left(n\frac{\pi}{2}\right)+(c+dn)\sin\left(n\frac{\pi}{2}\right)\right),\ a,b,c,d\in\mathbb{R}$.

1.1.7 $y(n+6)-2y(n+3)+2y(n)=0$

(i) $\chi(t)=t^6-2t^3+2$,

(ii) $\left\{\sqrt[6]{2}(\cos(\alpha)+i\sin(\alpha));\ \alpha\in\left\{\frac{\pi}{12},\frac{7\pi}{12},\frac{3\pi}{4},\frac{5\pi}{4},\frac{17\pi}{12},\frac{23\pi}{12}\right\}\right\}$,

(iii) $\left\{2^{\frac{n}{6}}\cos(\alpha),2^{\frac{n}{6}}\sin(\alpha);\ \alpha\in\left\{\frac{\pi}{12},\frac{7\pi}{12},\frac{3\pi}{4}\right\}\right\}$,

(iv) $y(n)=2^{\frac{n}{6}}\sum_{i=1}^{3}(a_i\cos(\alpha_i)+b_i\sin(\alpha_i)),\ a_1,a_2,a_3,b_1,b_2,b_3\in\mathbb{R},\ \alpha_1=\frac{\pi}{12}$, $\alpha_2=\frac{7\pi}{12},\ \alpha_3=\frac{3\pi}{4}$.

1.2 Rovnice se specialni pravou stranou

Nejprve resime prislusnou homogenni rovnici (viz predchozi kapitola). Pak nalezneme partikularni reseni. Nakonec nalezneme obecne reseni viz bod (C).

Postup nalezeni partikularniho a obecneho reseni:

(v) Nalezeni m, α, ν a $k = \max\{stP, stQ\}$, z vety o specialni prave strane (V5), kde prava strana je rovna $\alpha^n(P(n)\cos(n\nu) + Q(n)\sin(n\nu))$ a m je nasobnost cisla $\alpha(\cos(\nu) + i\sin(\nu))$ jakozto korene charakteristickeho polynomu $\chi(t)$ (viz bod (ii)).

(vi) Pomoci k vyjadrime obecne tvary polynomu R, S (napr.: $k = 2$ implikuje $R(n) = an^2 + bn + c$, kde $a, b, c \in \mathbb{R}$. Dosadime tyto obecne polynomy a drive nalezene m, α, ν do vety (V5) a obdrzime obecny tvar partikularniho reseni $y_p(n) = n^m\alpha^n(R(n)\cos(n\nu) + S(n)\sin(n\nu))$.

(vii) Dosadime partikularni reseni do rovnice a dopocitame presny tvar polynomu R a S a tim tez presny tvar $y_p(n)$. Take lze dosadit za n urcita cisla (napr. $0, 1, 2$ atd.) a obdrzet soustavu rovnic. Je potreba dosadit tolik cisel, aby bylo mozno soustavu rovnic vyresit.

(viii) Nalezneme obecne reseni viz bod (C). V pripade pocatecni podminky, nalezeni reseni splnujiciho pocatecni podminku.

1.2.1 $y(n+4) - y(n) = \sin\left(\frac{n\pi}{4}\right)$

(i) $\chi(t) = t^4 - 1$,

(ii) $\{\pm 1, \pm i\}$,

(iii) $\left\{1, (-1)^n, \sin\left(\frac{n\pi}{2}\right), \cos\left(\frac{n\pi}{2}\right)\right\}$,

(iv) $y_h(n) = A + B(-1)^n + C\sin\left(\frac{n\pi}{2}\right) + D\cos\left(\frac{n\pi}{2}\right)$, $A, B, C, D \in \mathbb{R}$.

(v) $m = 0, \alpha = 1, \nu = \frac{\pi}{4}, k = 0$.

(vi) $R(n) = a$, $S(n) = b$, $y_p(n) = a\cos\left(\frac{n\pi}{4}\right) + b\sin\left(\frac{n\pi}{4}\right)$, kde $a, b \in \mathbb{R}$.

(vii) $a = 0$, $b = -\frac{1}{2}$, $y_p(n) = -\frac{1}{2}\sin\left(\frac{n\pi}{4}\right)$.

(viii) $y(n) = -\frac{1}{2}\sin\left(\frac{n\pi}{4}\right) + A + B(-1)^n + C\sin\left(\frac{n\pi}{2}\right) + D\cos\left(\frac{n\pi}{2}\right)$, $A, B, C, D \in \mathbb{R}$.

1.2.2 $y(n+4) + y(n) = \sin\left(\frac{n\pi}{4}\right)$

(i) $\chi(t) = t^4 + 1$,

(ii) $\left\{\frac{1}{\sqrt{2}}(\pm 1 \pm i)\right\}$,

(iii) $\left\{\sin\left(\frac{n\pi}{4}\right), \cos\left(\frac{n\pi}{4}\right), \sin\left(\frac{3n\pi}{4}\right), \cos\left(\frac{3n\pi}{4}\right)\right\}$,

(iv) $y_h(n) = A\sin\left(\frac{n\pi}{4}\right) + B\cos\left(\frac{n\pi}{4}\right) + C\sin\left(\frac{3n\pi}{4}\right) + D\cos\left(\frac{3n\pi}{4}\right)$, $A, B, C, D \in \mathbb{R}$.

(v) $m = 1, \alpha = 1, \nu = \frac{\pi}{4}, k = 0$.

(vi) $R(n) = a$, $S(n) = b$, $y_p(n) = n(a\cos\left(\frac{n\pi}{4}\right) + b\sin\left(\frac{n\pi}{4}\right))$, kde $a, b \in \mathbb{R}$.

(vii) $a = 0$, $b = -\frac{1}{4}$, $y_p(n) = -\frac{1}{4}n\sin\left(\frac{n\pi}{4}\right)$.

(viii) $y(n) = -\frac{1}{4}n\sin\left(\frac{n\pi}{4}\right) + A\sin\left(\frac{n\pi}{4}\right) + B\cos\left(\frac{n\pi}{4}\right) + C\sin\left(\frac{3n\pi}{4}\right) + D\cos\left(\frac{3n\pi}{4}\right)$, $A, B, C, D \in \mathbb{R}$.

1.2.3 $y(n+2) - y(n+1) + y(n) = \sin\left(\frac{n\pi}{3}\right)$

(i) $\chi(t) = t^2 - t + 1$,

(ii) $\left\{\frac{1+i\sqrt{3}}{2}, \frac{1-i\sqrt{3}}{2}\right\}$,

(iii) $\left\{\cos\left(\frac{n\pi}{3}\right), \sin\left(\frac{n\pi}{3}\right)\right\}$,

(iv) $y_h(n) = A\cos\left(\frac{n\pi}{3}\right) + B\sin\left(\frac{n\pi}{3}\right),\ A, B \in \mathbb{R}$.

(v) $m = 1, \alpha = 1, \nu = \frac{\pi}{3}, k = 0$.

(vi) $R(n) = a,\ S(n) = b,\ y_p(n) = n\left(a\cos\left(\frac{n\pi}{3}\right) + b\sin\left(\frac{n\pi}{3}\right)\right)$, kde $a, b \in \mathbb{R}$.

(vii) $a = -\frac{1}{2\sqrt{3}},\ b = -\frac{1}{2},\ y_p(n) = -n\left(\frac{1}{2\sqrt{3}}\cos\left(\frac{n\pi}{3}\right) + \frac{1}{2}\sin\left(\frac{n\pi}{3}\right)\right)$.

(viii) $y(n) = -n\left(\frac{1}{2\sqrt{3}}\cos\left(\frac{n\pi}{3}\right) + \frac{1}{2}\sin\left(\frac{n\pi}{3}\right)\right) + A\cos\left(\frac{n\pi}{3}\right) + B\sin\left(\frac{n\pi}{3}\right),\ A, B \in \mathbb{R}$.

1.2.4 $y(n+2) - 2y(n+1) + 2y(n) = \cos(n)$

(i) $\chi(t) = t^2 - 2t + 2$,

(ii) $\{1+i, 1-i\}$,

(iii) $\left\{2^{\frac{n}{2}}\sin\left(\frac{n\pi}{4}\right), 2^{\frac{n}{2}}\cos\left(\frac{n\pi}{4}\right)\right\}$,

(iv) $y_h(n) = A2^{\frac{n}{2}}\sin\left(\frac{n\pi}{4}\right) + B2^{\frac{n}{2}}\cos\left(\frac{n\pi}{4}\right),\ A, B \in \mathbb{R}$.

(v) $m = 0, \alpha = 1, \nu = 1, k = 0$.

(vi) $R(n) = a,\ S(n) = b,\ y_p(n) = a\cos(n) + b\sin(n)$, kde $a, b \in \mathbb{R}$.

(vii) $a = \frac{\cos(2)-2\cos(1)+2}{9-12\cos(1)+4\cos(2)},\ b = \frac{\sin(2)-2\sin(1)}{9-12\cos(1)+4\cos(2)}$.

(viii) $y(n) = y_p(n) + A2^{\frac{n}{2}}\sin\left(\frac{n\pi}{4}\right) + B2^{\frac{n}{2}}\cos\left(\frac{n\pi}{4}\right),\ A, B \in \mathbb{R}$.

1.2.5 $y(n+2) - 3y(n+1) + 2y(n) = n^2$, $y(1) = 3$, $y(2) = 2$

(i) $\chi(t) = t^2 - 3t + 2$,

(ii) $\{1, 2\}$,

(iii) $\{1, 2^n\}$,

(iv) $y_h(n) = A + B2^n,\ A, B \in \mathbb{R}$.

(v) $m = 1, \alpha = 1, \nu = 0, k = 2$.

(vi) $R(n) = an^2 + bn + c,\ y_p(n) = n\left(an^2 + bn + c\right)$, kde $a, b, c \in \mathbb{R}$.

(vii) $a = -\frac{1}{3},\ b = -\frac{1}{2},\ c = -\frac{13}{6}$.

(viii) $y(n) = B2^n - \frac{1}{3}n^3 - \frac{1}{2}n^2 - \frac{13}{6}n + A,\ A = 1,\ B = \frac{5}{2}$.

1.2.6 $y(n+2) - y(n) = 17$, $y(1) = y(2) = 0$

(i) $\chi(t) = t^2 - 1$,

(ii) $\{-1, 1\}$,

(iii) $\{(-1)^n, 1\}$,

(iv) $y_h(n) = A(-1)^n + B$, $A, B \in \mathbb{R}$.

(v) $m = 1, \alpha = 1, \nu = 0, k = 0$.

(vi) $R(n) = a$, $y_p(n) = an$, kde $a \in \mathbb{R}$.

(vii) $a = \frac{17}{2}$.

(viii) $y(n) = A(-1)^n + \frac{17}{2}n + B$, $A = -\frac{17}{4}$, $B = -\frac{51}{4}$.

1.3 Rovnice s pravou stranou ve tvaru souctu specialnich pravych stran

V nekterych pripadech nema prava strana specialni tvar, ale ma tvar souctu vice specialnich pravych stran. Tedy, $PS = \sum_{i=1}^{s} f_i(n)$, kde $f_i(n)$ ma specialni tvar (viz veta V5) pro $i = 1, \ldots, s$. V takovemto pripade krome homogenniho reseni $y_h(n)$ spocitame s partikularnich reseni $y_p^i(n)$, $i = 1, \ldots, s$, ktera budou odpovidat prislusnym pravym stranam. Obecne reseni pak bude mit tvar $y(n) = y_h(n) + \sum_{i=1}^{s} y_p^i(n)$.

1.3.1 $y(n+3) - y(n+2) - 2y(n+1) + 2y(n) = n + 2^n$

(i) $\chi(t) = t^3 - t^2 - 2t + 2$,

(ii) $\{1, \pm\sqrt{2}\}$,

(iii) $\left\{1, (-\sqrt{2})^n, 2^{\frac{n}{2}}\right\}$,

(iv) $y_h(n) = A + B(-\sqrt{2})^n + C2^{\frac{n}{2}}$, $A, B, C \in \mathbb{R}$.

$f_1(n) = n :$

(v)$_1$ $m = 1, \alpha = 1, \nu = 0, k = 1$.

(vi)$_1$ $R(n) = an + b$, $y_p^1(n) = n(an + b)$, kde $a, b \in \mathbb{R}$.

(vii)$_1$ $a = -\frac{1}{2}$, $b = -\frac{3}{2}$.

$f_2(n) = 2^n :$

(v)$_2$ $m = 0, \alpha = 2, \nu = 0, k = 0$.

(vii)$_2$ $R(n) = a$, $y_p^2(n) = a2^n$, kde $a \in \mathbb{R}$.

(vii)$_2$ $a = \frac{1}{2}$.

(viii) $y(n) = y_h(n) + y_p^1(n) + y_p^2(n) = A + B(-\sqrt{2})^n + C2^{\frac{n}{2}} - \frac{1}{2}n^2 - \frac{3}{2}n + 2^{n-1}$.

1.3.2 $8y(n+3)+y(n)=3n+2^{-n}$

(i) $\chi(t)=8t^3+1$,

(ii) $\left\{-\frac{1}{2},\frac{1}{2}\left(\cos\left(\pm\frac{\pi}{3}\right)+i\sin\left(\pm\frac{\pi}{3}\right)\right)\right\}$,

(iii) $\left\{\left(-\frac{1}{2}\right)^n,2^{-n}\cos\left(\pm\frac{n\pi}{3}\right),2^{-n}\sin\left(\pm\frac{n\pi}{3}\right)\right\}$,

(iv) $y_h(n)=A\left(-\frac{1}{2}\right)^n+B2^{-n}\cos\left(\pm\frac{n\pi}{3}\right)+C2^{-n}\sin\left(\pm\frac{n\pi}{3}\right)$, $A,B,C\in\mathbb{R}$.

$f_1(n)=3n$:

(v)$_1$ $m=0,\alpha=1,\nu=0,k=1$.

(vi)$_1$ $R(n)=an+b$, $y_p^1(n)=an+b$, kde $a,b\in\mathbb{R}$.

(vii)$_1$ $a=\frac{1}{3}$, $b=-\frac{8}{9}$.

$f_2(n)=2^{-n}$:

(v)$_2$ $m=0,\alpha=\frac{1}{2},\nu=0,k=0$.

(vi)$_2$ $R(n)=a$, $y_p^2(n)=a2^{-n}$, kde $a\in\mathbb{R}$.

(vii)$_2$ $a=\frac{1}{2}$.

(viii) $y(n)=y_h(n)+y_p^1(n)+y_p^2(n)=A\left(-\frac{1}{2}\right)^n+B2^{-n}\cos\left(\pm\frac{n\pi}{3}\right)+C2^{-n}\sin\left(\pm\frac{n\pi}{3}\right)+\frac{1}{3}n-\frac{8}{9}+2^{-n-1}$.